

Instability of high-frequency acoustic waves in accretion disks with turbulent viscosity

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Abstract. Dynamics of linear perturbations in a differentially rotating accretion disk with non-homogeneous vertical structure is investigated. It has been found that turbulent viscosity results in instability of both pinching oscillations, and bending modes. Not only the low-frequency fundamental modes, but also the high-frequency reflective harmonics appear to be unstable. The question of the limits of applicability of the thin disk model (MTD) is also investigated. The insignificant distinctions in the dispersion properties of MTD and three-dimensional model appear for wave numbers $k \lesssim (1 \div 3)/h$ (h is the half-thickness of a disk). In the long-wavelength limit, the relative difference between eigenfrequencies of the unstable acoustic mode in the 3D-model and the MTD is smaller than 5 %.

Key words: accretion disk — instability — turbulent viscosity — acoustic mode

1. Introduction

Accretion disks (AD) are important in many observable astrophysical objects (close binary systems, quasars, active galactic nuclei, young stars, protoplanet disks). It is usually assumed that the disk is geometrically thin, and that the disk has turbulent viscosity (Shakura & Sunyaev 1973; Lightman & Eardley 1974). The physical mechanism of turbulent viscosity can be connected with various instabilities.

The basis for various viscous models of AD is the assumption, that the dynamic viscosity η is caused by turbulence of medium and that $\eta \sim \rho u_t \ell_t$ (u_t is the characteristic amplitude of the most large-scale turbulent velocity, ℓ_t is the spatial scale, ρ is the density) (Shakura & Sunyaev 1973). In this connection the vital importance is given to the research of multimode instabilities, which result in a complex perturbations structure. Different spatial and temporal scales may lead to the development of turbulence in the disk.

The thin disk model is the widely used for studies of the accretion disk dynamics. The MTD involves averaging over z -coordinate of the three-dimensional hydrodynamic equations, if a number of additional conditions is fulfilled (Shakura & Sunyaev 1973; Gor'kavyi & Fridman 1994; Khoperskov & Khrapov 1995). In the context of two-dimensional (in the plane of the disk) models without magnetic field, four unstable modes of oscillations are present: two acoustic modes (Wallinder 1991; Wu & Yang 1994; Khoperskov & Khrapov 1995; Wu et al. 1995), a thermal mode and a viscous one (Lightman & Eardley 1974; Shakura & Sunyaev 1976; Szuszkiewicz 1990; Wallinder 1991; Wu & Yang 1994). The instability growth rate of acoustic waves increases with the reduction of wavelength λ . However, the thin disk model imposes the restriction on wavelength, $\lambda \gg h$, and therefore it is necessary to consider AD z -structure for a correct treatment when $\lambda \lesssim h$. The thin disk model applies only to pinching oscillations. Then the perturbed pressure is a symmetric function, and the displacement of gas does not move the mass centre in a disk with respect to a symmetry plane ($z = 0$). Thus, bending oscillations (AS-mode) are excluded from consideration, for AS-mode the perturbed pressure is antisymmetric function. Also, in two-dimensional models the high-frequency (reflective) harmonics with characteristic spatial scales in z -direction $\lesssim (0.5 \div 1)h$ cannot be investigated.

In this paper we investigate the dynamics of acoustic perturbations taking into account the non-homogeneous z -structure of a viscous disk. Apart from the special problem of the limits of MTD applicability, main question is the existence of instabilities in short-wave region and, in addition, the stability of high-frequency harmonics. In Sect. 2 we define the AD model and we choose the viscosity law. In Sect. 3 we consider the dynamics of linear acoustic perturbations, and formulate a mathematical problem of eigenfrequency determination for various unstable modes in the disk. Lastly in Sects. 4 and 5 we discuss results of the numerical solution of the boundary problem and summarize the main conclusions.

2. Model and basic equations

We shall consider an axisymmetric differentially rotating gas disk in the gravitational field of a mass M . Without including self-gravity and relativistic effects, and adopting cylindrical coordinates we have:

$$\Psi(r, z) = -\frac{GM}{(r^2 + z^2)^{1/2}} \simeq -\frac{GM}{r} + \frac{1}{2}\Omega_k^2 z^2, \quad (1)$$

G is the gravitational constant, $\Omega_k = \sqrt{GM/r^3}$ is Keplerian angular velocity.

We shall use the axisymmetric hydrodynamic equations in view of viscosity. The equations of motion and continuity have the form

$$\frac{du}{dt} - \frac{v^2}{r} = \frac{1}{\rho} \left(-\frac{\partial P}{\partial r} + \frac{\partial r \sigma_{rr}}{r \partial r} - \frac{\sigma_{\varphi\varphi}}{r} + \frac{\partial \sigma_{rz}}{\partial z} \right) - \frac{\partial \Psi}{\partial r}, \quad (2)$$

$$\frac{dv}{dt} + \frac{uv}{r} = \frac{1}{\rho} \left(\frac{\partial \sigma_{\varphi z}}{\partial z} + \frac{\partial r^2 \sigma_{r\varphi}}{r^2 \partial r} \right), \quad (3)$$

$$\frac{dw}{dt} = \frac{1}{\rho} \left(-\frac{\partial P}{\partial z} + \frac{\partial r \sigma_{rz}}{r \partial r} + \frac{\partial \sigma_{zz}}{\partial z} \right) - \frac{\partial \Psi}{\partial z}, \quad (4)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(r\rho u)}{r \partial r} + \frac{\partial(\rho w)}{\partial z} = 0, \quad (5)$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}$, $\mathbf{V} = (u, v, w)$ is the velocity, P is the pressure, ρ is the volume density of matter in a disk, σ_{ij} is the components of symmetric viscous stress tensor ($\sigma_{ij} = \sigma_{ji}$).

We shall add the thermal equation to the system of Eqs. (2)–(5) as

$$\frac{dS}{dt} = \frac{Q}{T}, \quad (6)$$

where S is the entropy, T is the temperature, and the variable Q defines sources of heat.

2.1. Equilibrium model and viscosity law

We shall assume that the equilibrium velocity in the disk has only r and φ components: $\mathbf{V}_0 = (U_0, V_0, 0)$. The expression for components of the viscous stress tensor may be written in the following form:

$$\sigma_{ij} = -\alpha_{ij} P, \quad \alpha_{ij} = \text{const} > 0, \quad (7)$$

where $i, j = (r, \varphi, z)$. As without the account of the second (volume) viscosity the trace of the viscous tensor is equal to zero ($\delta_{ij} \sigma_{ji} = 0$), thus $\alpha_{ii} = 0$ ($\sigma_{rr} = \sigma_{\varphi\varphi} = \sigma_{zz} = 0$).

The parameters α_{rz} , $\alpha_{\varphi z}$ and $\alpha_{r\varphi}$ determine a level of turbulence in the disk. Moreover α_{rz} and $\alpha_{\varphi z}$ are caused by a shear character of flow in z -direction, and the value $\alpha_{r\varphi}$ is connected to differentiability of rotation in the disk

plane and coincides with α -parameter of the standard theory of the disk accretion (Shakura & Sunyaev 1973). For α_{ij} it is possible to write down:

$$\alpha_{r\varphi} = -\frac{\eta_0}{P_0} l_\Omega \Omega, \alpha_{rz} = -\frac{\eta_0}{P_0} \frac{\partial U_0}{\partial z}, \alpha_{\varphi z} = -\frac{\eta_0}{P_0} \frac{\partial V_0}{\partial z}, \quad (8)$$

here $l_\Omega = \partial(\ln \Omega)/\partial(\ln r)$. One must be limited by the case of weak dependence of equilibrium velocities (U_0 and V_0) on z -coordinate for correct transition to the thin disk model. This is possible if the following conditions: $\alpha_{rz} \ll \alpha$ and $\alpha_{\varphi z} \ll \alpha$ are fulfilled. Therefore it is possible not to take into account in the equations the terms, containing σ_{rz} and $\sigma_{\varphi z}$. The conditions $U_0 \simeq \alpha(h/r)^2 V_0$ and $h/r \ll 1$ are carried out for the majority of the models of accretion disks. In view of the made above assumptions the equilibrium balance of the forces is defined by the equations:

$$\frac{V_0^2}{r} = \frac{1}{\rho_0} \frac{\partial P_0}{\partial r} + \frac{\partial \Psi}{\partial r}, \quad (9)$$

$$U_0 \frac{\partial r V_0}{r \partial r} = -\frac{\alpha}{\rho_0} \frac{\partial r^2 P_0}{r^2 \partial r}, \quad (10)$$

$$\frac{\partial P_0}{\partial z} = -\rho_0 \frac{\partial \Psi}{\partial z}. \quad (11)$$

The equilibrium functions can be submitted in the self-similar form: $f_0(r, z) = f_{01}(r) f_{02}(z)$, if it is assumed $h/r \ll 1$. The disk z -structure depends on the state equation and the energy flux. For the sake of simplicity we shall limit ourselves to a polytropic model

$$\frac{\partial}{\partial z} \left\{ \frac{P_0(r, z)}{[\rho_0(r, z)]^n} \right\} = 0. \quad (12)$$

Then for the pressure and density it is possible to write the solutions:

$$P_0(r, z) = P_0(r, 0) F(z)^a, \quad \rho_0(r, z) = \rho_0(r, 0) F(z)^b, \quad (13)$$

where $F(z) = 1 - z^2/h^2$, $a = n/(n-1)$, $b = 1/(n-1)$, n is polytropic index, and h defines the disk boundary — in the points $z = \pm h$ the equilibrium pressure and density are equal to zero. We assume below that $h(r) = \text{const}$.

The relation

$$C_s^2(r, z) = \frac{\gamma P_0}{\rho_0} = C_s^2(r, 0) F(z), \quad C_s^2(r, 0) = \frac{\gamma}{a} \Omega_k^2 h^2, \quad (14)$$

defines the adiabatic sound speed (γ is adiabatic index). For the equilibrium velocities (V_0 and U_0) from Eqs. (9), (10) taking into account Eqs. (13) and (14) we obtain:

$$V_0(r, z) = r \Omega(r, z) = \sqrt{r^2 \Omega_k^2 + l_P C_s^2 / \gamma}, \quad (15)$$

$$U_0(r, z) = -\frac{\alpha(2 + l_P)}{\gamma(1 + l_V)} \frac{C_s^2}{V_0}, \quad (16)$$

where $l_P = \partial(\ln P_0)/\partial(\ln r)$, $l_\rho = \partial(\ln \rho_0)/\partial(\ln r)$, $l_V = \partial(\ln V_0)/\partial(\ln r)$.

3. Linear analysis

In the framework of standard linear analysis the pressure, density and velocity are represented as:

$$u = U_0(r, z) + \tilde{u}(r, z, t),$$

$$v = V_0(r, z) + \tilde{v}(r, z, t),$$

$$w = \tilde{w}(r, z, t),$$

$$P = P_0(r, z) + \tilde{P}(r, z, t),$$

$$\rho = \rho_0(r, z) + \tilde{\rho}(r, z, t).$$

In the linear approximation ($|\tilde{f}| \ll |f_0|$) from Eqs. (1)–(5) the linearized equations become:

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} + U_0 \frac{\partial \tilde{u}}{\partial r} + \tilde{u} \frac{\partial U_0}{\partial r} + \tilde{w} \frac{\partial U_0}{\partial z} - \frac{2V_0}{r} \tilde{v} = \\ - \frac{1}{\rho_0} \left(\frac{\partial \tilde{P}}{\partial r} - \frac{\tilde{\rho}}{\rho_0} \frac{\partial P_0}{\partial r} \right), \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t} + U_0 \frac{\partial(r\tilde{v})}{r\partial r} + \tilde{u} \frac{\partial(rV_0)}{r\partial r} + \tilde{w} \frac{\partial V_0}{\partial z} = \\ - \frac{\alpha}{\rho_0} \left(\frac{\partial(r^2\tilde{P})}{r^2\partial r} - \frac{\tilde{\rho}}{\rho_0} \frac{\partial(r^2P_0)}{r^2\partial r} \right), \end{aligned} \quad (18)$$

$$\frac{\partial \tilde{w}}{\partial t} + U_0 \frac{\partial \tilde{w}}{\partial r} = - \frac{\partial \tilde{P}}{\rho_0 \partial z} - g \frac{\tilde{\rho}}{\rho_0}, \quad (19)$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial}{r\partial r} [r(U_0\tilde{\rho} + \rho_0\tilde{u})] + \frac{\partial(\rho_0\tilde{w})}{\partial z} = 0, \quad (20)$$

where $g \equiv \partial\Psi/\partial z$.

As we shall study the dynamics of acoustic oscillations, it is possible to set $Q = 0$. Then the thermal Eq. (6) becomes:

$$\frac{\partial \tilde{S}}{\partial t} + U_0 \frac{\partial \tilde{S}}{\partial r} + \tilde{u} \frac{\partial S_0}{\partial r} + \tilde{w} \frac{\partial S_0}{\partial z} = 0, \quad (21)$$

here $S_0 = c_V \ln(P_0/\rho_0^{\gamma})$ is the entropy of equilibrium gas. We shall exclude \tilde{S} from Eq. (21) with the help of the equation of state $\tilde{S} = \tilde{S}(\tilde{P}, \tilde{\rho})$. In the linear approximation we obtain:

$$\tilde{S} = \left(\frac{\partial S}{\partial P} \right)_\rho \tilde{P} + \left(\frac{\partial S}{\partial \rho} \right)_P \tilde{\rho} = c_V \frac{\tilde{P}}{P_0} - c_P \frac{\tilde{\rho}}{\rho_0}, \quad (22)$$

where $c_V = T(\partial S/\partial T)_\rho$, $c_P = T(\partial S/\partial T)_P$ are specific heats at constant density and pressure.

Short-wave approximation in a radial direction ($kr \gg 1$, k is radial wave number) makes it possible to write the solution as:

$$\tilde{f}(r, z, t) = \hat{f}(z) \exp\{ikr - i\omega t\}, \quad (23)$$

here ω is complex eigenfrequency.

Taking into account Eqs. (22) and (23), the system Eqs. (17)–(21) is reduced to two ordinary differential equations for amplitudes of the perturbed pressure $\hat{P}(z)$ and

the material displacement $\hat{\xi}(z)$ from an equilibrium position:

$$\frac{d\hat{\xi}}{dz} = \frac{D_1}{\hat{\omega}q^2C_s^2} \frac{\hat{P}}{\rho_0} + \left\{ \frac{kD_2}{\hat{\omega}q^2} + \frac{g}{C_s^2} \right\} \hat{\xi}, \quad (24)$$

$$\frac{d\hat{P}}{dz} = \rho_0 \left\{ \hat{\omega}^2 - \frac{g}{\gamma} D_3 \right\} \hat{\xi} - \frac{g}{C_s^2} \{1 + D_4\} \hat{P}, \quad (25)$$

where $\hat{\omega} = \omega - kU_0$, $q^2 = \hat{\omega}^2 - \alpha^2$, $\alpha = \Omega\sqrt{2(2 + l_\Omega)}$ is the epicyclic frequency, $D_1 = -\hat{\omega}^3 + \hat{\omega}(\alpha^2 + k^2C_s^2) + 2i\alpha\Omega k^2C_s^2$,

$$D_2 = i\omega F_V - 2\Omega F_U, \quad D_3 = S_z - \frac{iS_r}{\hat{\omega}q^2} (\hat{\omega}F_U + 2i\Omega F_V),$$

$$D_4 = \frac{ikS_r}{\gamma\hat{\omega}q^2} (\hat{\omega} + 2i\alpha\Omega), \quad F_V = \frac{l_P C_s^2}{r\gamma^2} S_z - 2\Omega V_0', \quad F_U = \frac{\alpha(2 + l_P)C_s^2}{r\gamma^2} S_z + \frac{\alpha^2}{2\Omega} U_0', \quad S_z = \frac{1}{c_V} \frac{\partial S_0}{\partial z} = (\ln P_0)' -$$

$\gamma(\ln \rho_0)'$, $S_r = \frac{1}{c_V} \frac{\partial S_0}{\partial r} = (l_P - \gamma l_\rho)/r$, the stroke means differentiation with respect to z -coordinate ($f' \equiv \partial f/\partial z$), $\hat{\xi}$ is the complex amplitude of material z -displacement from equilibrium state, and the following is true $\tilde{w} = d\hat{\xi}/dt = -i\hat{\omega} \hat{\xi}$.

Boundary conditions must be added to Eqs. (24) and (25). In view of the symmetry, it is natural to consider two types of oscillation: 1) the symmetric oscillations ($\hat{\xi}(z) = -\hat{\xi}(-z)$ or $\hat{P}(z) = \hat{P}(-z)$), and therefore

$$\hat{\xi}(0) = 0 \quad \text{or} \quad \left. \frac{d\hat{P}}{dz} \right|_{z=0} = 0, \quad (26)$$

(such oscillations correspond to the pinch-mode or S-mode); 2) the antisymmetric oscillations ($\hat{\xi}(z) = \hat{\xi}(-z)$ or $\hat{P}(z) = -\hat{P}(-z)$), and therefore

$$\hat{P}(0) = 0 \quad \text{or} \quad \left. \frac{d\hat{\xi}}{dz} \right|_{z=0} = 0, \quad (27)$$

that correspond to the bending oscillations or AS-mode. On the disk unperturbed surface the following condition should be fulfilled:

$$\hat{P}(h) + \left. \frac{\partial P_0}{\partial z} \right|_{z=h} \hat{\xi}(h) = 0. \quad (28)$$

By solving the system of Eqs. (24), (25) with the boundary conditions Eqs. (28) and (26) (or (27)) (boundary problem), we find the eigenvalues of complex frequency ω for the given distribution of equilibrium parameters along the vertical coordinate in the disk. A positive imaginary part of the frequency (growth rate) means that the eigen-mode is unstable. The above-stated outline will have the name "3D-model", and the model of thin disk — "2D-model".

3.1. Dispersion relation

For a homogeneous z -distribution of the equilibrium quantities, the system of Eqs. (24) and (25) is reduced to the following dispersion relation:

$$\omega^4 - \omega^2[\alpha^2 + C_s^2(k^2 + k_z^2)] - 2i\omega\alpha\Omega C_s^2 k^2 + \alpha^2 C_s^2 k_z^2 = 0, \quad (29)$$

here k_z is the wave number in z -direction. It is necessary to stress that Eq. (29) at $k_z = 0$ coincides with the dispersion relation earlier obtained for 2D-model in case of replacement γ by the flat adiabatic index $\Gamma = \Gamma_1 = (3\gamma - 1)/(\gamma + 1)$ (Wallinder 1991; Khoperskov & Khrapov 1995). Kovalenko and Lukin (1998) have obtained the more accurate formula for the flat adiabatic index $\Gamma = \Gamma_2 = (3\gamma - 1 - \gamma\delta)/(\gamma + 1 - \delta)$ ($\delta = \alpha^2/\Omega_k^2$). This relation takes into account Keplerian rotation of a thin disk. The adiabatic sound speed in the disk plane should be determined by the following expression:

$$C_s^2 = \Gamma \frac{\int_0^h P(z) dz}{\int_0^h \rho(z) dz}. \quad (30)$$

Thus, taking into account Eq. (30), the dispersion relation Eq. (29) in the case of $k_z = 0$ describes dynamics of perturbations within the limits of flat model (MTD). Obviously, these oscillations correspond to S-mode.

In the general case when $\alpha > 0$, this dispersion relation gives two unstable acoustic branches of oscillations and two damping modes (viscous and thermal ones). The damping of the viscous or thermal modes depends on the fact that we don't take into account dissipation and radiative processes in the thermal equation. If these factors are taken into account, then the viscous and thermal low-frequency oscillatory branches can be unstable (Shakura & Sunyaev 1976), but this effect will not change the dispersion properties of sound waves.

4. Results

It is convenient to characterize the properties of the considered acoustic oscillations in the disk by the dimensionless frequency $W = \omega/\Omega$ and the dimensionless wave number kh . We shall define the basic model as follows: $\alpha = 0.2$, $\gamma = 5/3$, $h/r = 0.05$, $l_P = -3/2$, $l_\rho = -1/2$, $l_V = -1/2$ ($l_\Omega = -3/2$), $n = 5/3$. If it is not specified otherwise, the parameters take these values.

4.1. Fundamental S-mode in the 2D- and 3D-models

Our analysis confirms the good agreement between the thin disk model and the results of the 3D-problem solution in the case of the dissipational acoustic instability. The eigenfrequencies of oscillations obtained from Eq. (29) at $k_z = 0$ by taking into account Eq. (30) and from the solution of Eqs. (24), (25) practically coincide in the range $kh \lesssim 3$ (see Fig. 1a,b). The visible differences occur when $kh \gtrsim 3$. We emphasize that in this region the formal condition of applicability of the MTD ($kh \ll 1$) is obviously broken. In Fig. 1e,f, the dependencies of group velocity $v_g = \text{Re}(\partial\omega/\partial k)$ and $\text{Im}(\partial\omega/\partial k)$ on the wave number k for the 3D-model and the 2D-model at $\Gamma = \Gamma_1$ and $\Gamma = \Gamma_2$ are shown. In the case of small values of n , the divergence

occur at large kh as Fig. 1a,b shows. The reason is that the characteristic scale of inhomogeneity in the vertical direction increases with the reduction of n . The growth rate and the phase velocity of perturbations in the framework of the 3D-model is less than for the thin disk model (2D-model) in the short-wavelength region. This effect is caused by the inhomogeneous distribution of the equilibrium quantities in z -direction and by the transverse gravitational force. When the parameter α increases the growth rates linearly grow, and the value of the wave number k , at which the distinctions between the models appear, does not depend on α . The considered low-frequency mode exists both in the 2D-model and in the 3D-model, and it weakly depends on z -structure. We therefore call it the *fundamental mode*. In the case $\alpha \ll 1$, the expression for frequency is easily obtained from Eq. (29) as:

$$\omega = \pm \sqrt{\alpha^2 + k^2 C_s^2} + i\alpha \frac{\Omega C_s^2 k^2}{\alpha^2 + C_s^2 k^2}.$$

This approximation is exact enough. Our results shows, that the MTD adequately describes dynamics of perturbations with characteristic spatial scale $\lambda \sim 2h \div r$ in the disk plane.

The differential rotation ($l_\Omega = -\frac{d \ln \Omega}{d \ln r} > 0$) and a dependence of dynamic viscosity η on thermodynamic parameters (for example, dependence on density and temperature) are responsible for the instability. A simple model can demonstrate this effect. The last term in Eq. (3) (i.e., containing $\sigma_{r\varphi}$) is a source of the instability of acoustic modes. The expression $\eta = \sigma\nu$ (σ is the surface density, ν is the kinematic viscosity) is usually used in the 2D-model and for the sake of simplicity we assume here that ν is constant. Then only a perturbation of surface density $\tilde{\sigma}$ generates perturbation of dynamic viscosity $\tilde{\eta}$. Now let us consider how a viscous force, which is caused by $\tilde{\eta}$, leads to an amplification of the amplitude of sound wave in a disk plane. The evolution equation for the surface density, without including Coriolis's force, is $\frac{\partial}{\partial t} \left[\frac{\partial^2 \tilde{\sigma}}{\partial t^2} - C_s^2 \frac{\partial^2 \tilde{\sigma}}{\partial r^2} \right] = A \frac{\partial^2 \tilde{\sigma}}{\partial r^2}$, where $A = 2\Omega^2 l_\Omega \nu_0 > 0$. This equation is linear and the appropriate dispersion relation is $\omega[\omega^2 - C_s^2 k^2] = iAk^2$, which means that acoustic waves are unstable with the growth rate $\text{Im}(\omega) \simeq \frac{i}{2} \frac{A}{C_s^2}$ and entropy oscillations damping ($\text{Im}(\omega) \simeq -iA/C_s^2 < 0$). These estimates can only be used in the short-wavelength limit, because we have neglected epicyclic oscillations.

4.2. Fundamental AS-mode

We shall consider low-frequency bending oscillations for which Eq. (27) is fulfilled. In a longwavelength limit both symmetric and antisymmetric perturbations have identical frequency $\omega^2 \simeq \alpha^2 = \Omega_k^2$ in the case of Keplerian disk. This result can easily be obtained from Eq.

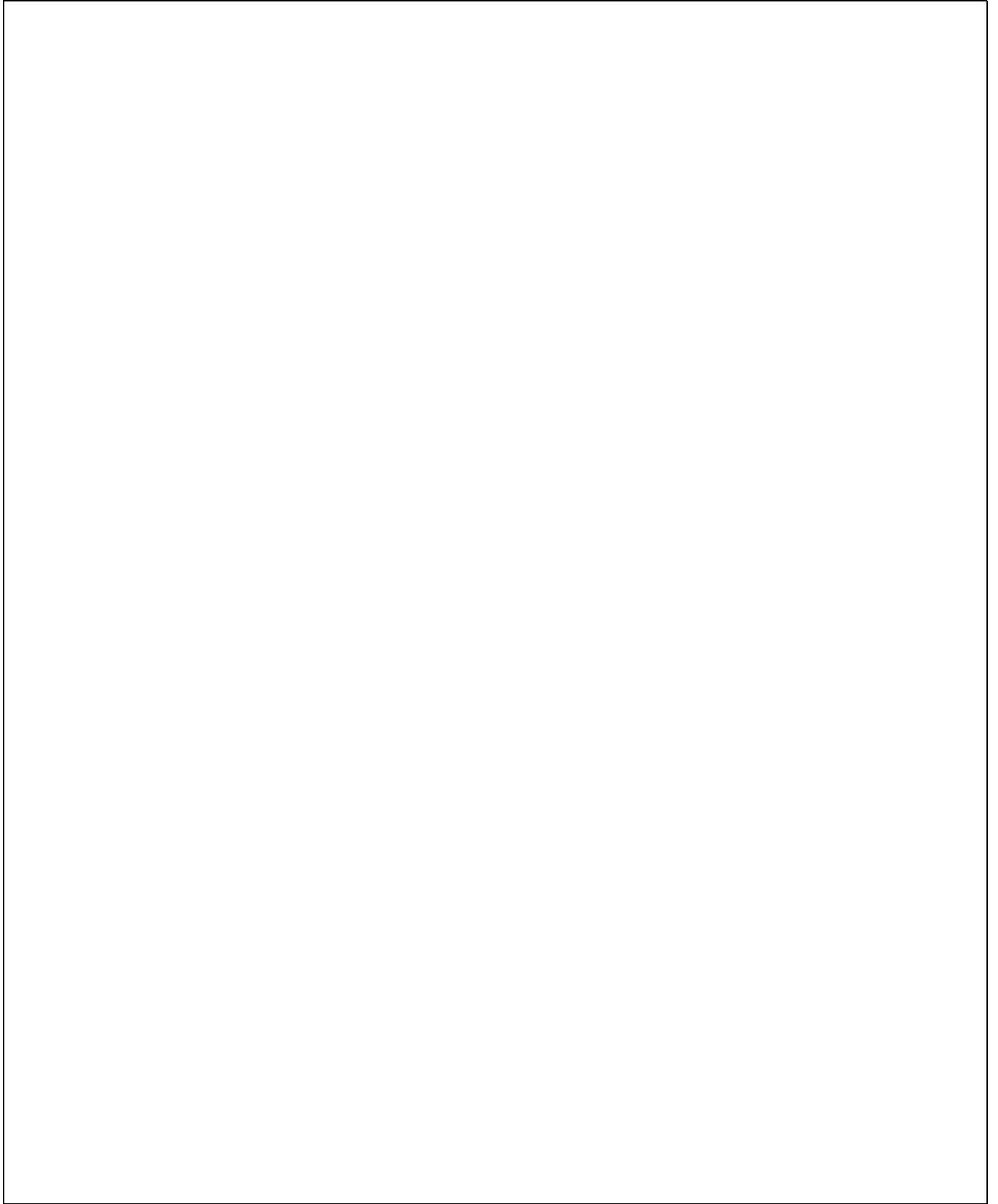


Fig. 1a-f. The dependence of the dimensionless eigenfrequency W on the dimensionless wave number kh for different values of n (1 — $n = 5/3$, 2 — $n = 1.2$) (see **a-d**). The dependence of $\partial W / \partial k$ on kh for S-mode (see **e,f**). The solid lines correspond to 3D-model, dotted lines — 2D-model at $\Gamma = \Gamma_1$, long-dashed lines — 2D-model at $\Gamma = \Gamma_2$.

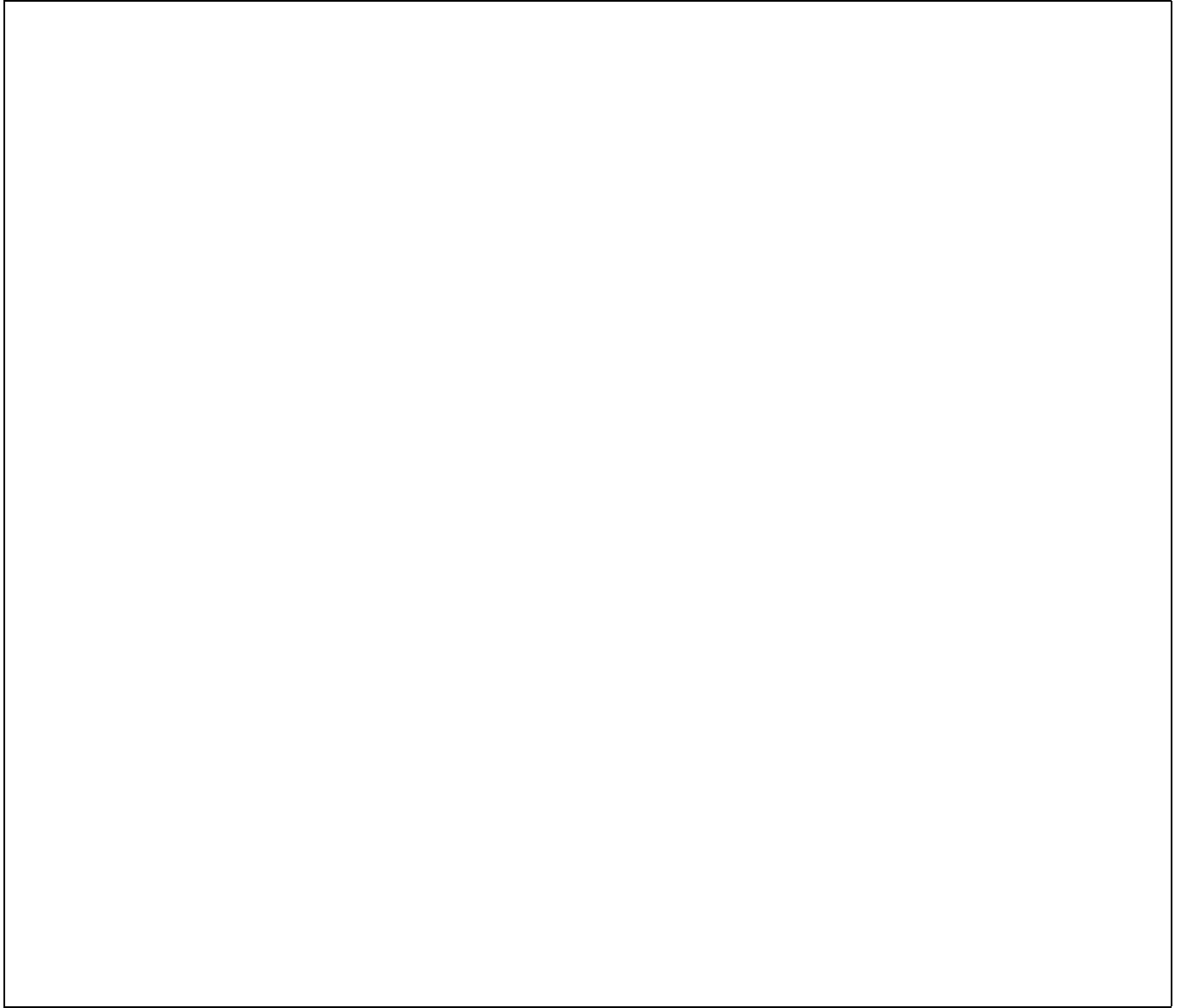


Fig. 2a-d. The dependence of W on kh for S- and AS-modes for various values of α . The solid line corresponds to $\alpha = 0.2$, the dotted line — $\alpha = 0.1$. The numbers of harmonics j are specified in Figure.

(29) when $k = 0$. In case of the adiabatic model two other branches of oscillations ($\omega^2 = (C_s k_z)^2$) always damp at $k > 0$ and we do not examine them. The dispersion behaviour of S- and AS-modes is very similar (Fig. 1a-d, solid lines). For the fundamental bending mode it is possible to set $k_z h \simeq \pi/2$ in Eq. (29) and for this case the dispersion curves on Fig. 1c,d are shown as dotted lines. The exact solution in the context of 3D-model shows visible difference from the result obtained from Eq. (29). In the range $kh \lesssim 0.1$ these differences are caused by the following factors: 1) the 3D-model includes radial inhomogeneities of equilibrium quantities; 2) the Eq. (29) does not take into account vertical inhomogeneity of disk. The obtained result is very unexpected and significant, since we used Eq. (29) beyond the limit of 2D-model approach.

The physical reason for AS-mode instability is similar to the case of S-oscillations.

4.3. High-frequency S- and AS-modes

Besides the fundamental S-mode, any number of unstable harmonics can be generated in addition. These harmonics differ from each other by the number of nodes of the perturbed pressure across the disk plane. The following estimate of the effective wave number in the z -direction can be made:

$$k_z h \simeq \pi j \text{ (S - mode)} \quad k_z h \simeq \pi(j + 1/2) \text{ (AS - mode)} ,$$

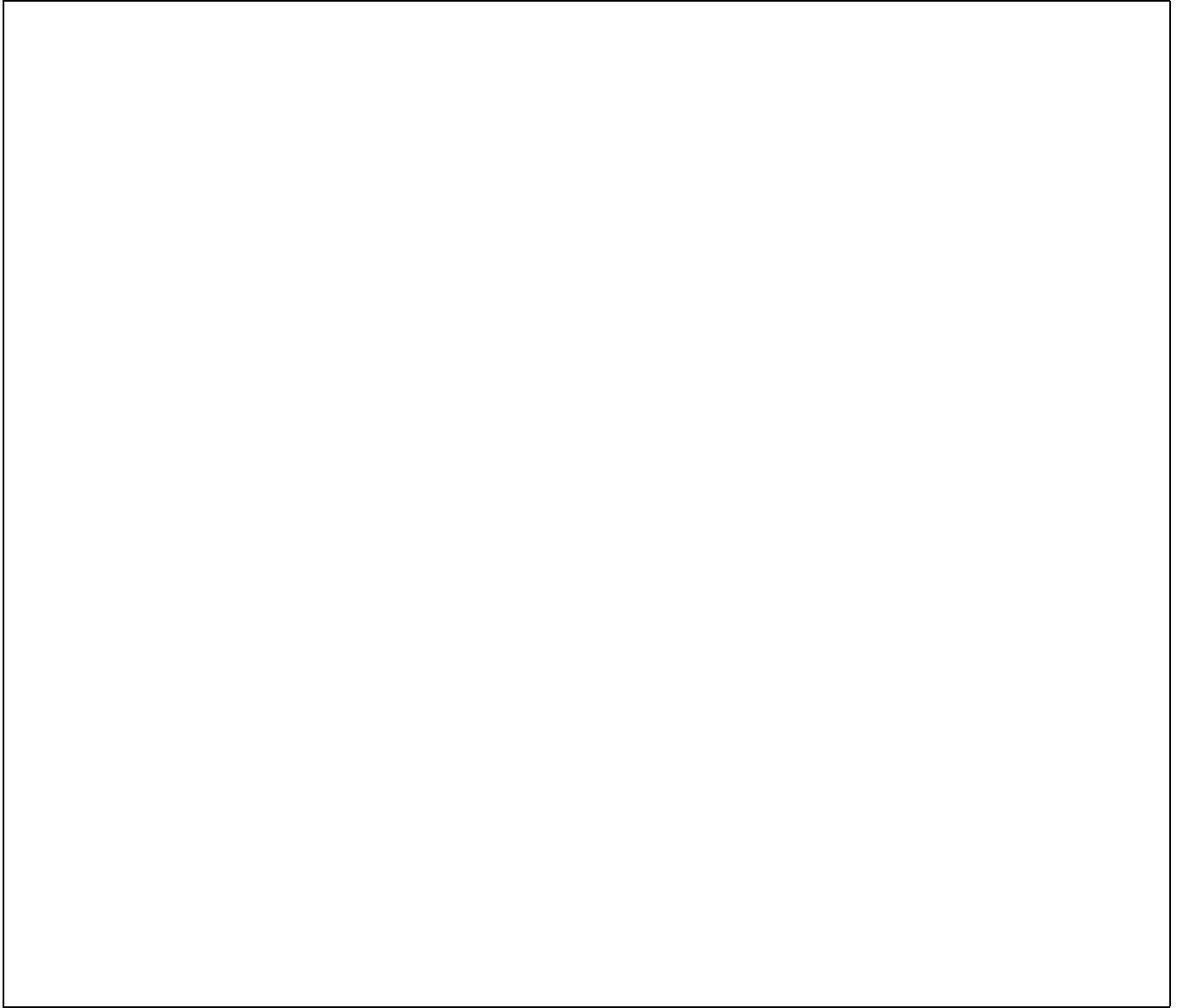


Fig. 3a-d. The dependence of W on kh for fundamental mode ($j = 0$) and for high-frequency one with $j = 3$ for various values of n . Solid line — $n = 5/3$, dotted line — $n = 1.2$, dashed line — 10 . The numbers of harmonics j are specified in Figure.

where j is the number of the harmonic. The fundamental ($j = 0$) and reflective ($j > 0$) harmonics exist for both symmetric (S-) and antisymmetric (AS-) modes.

The dependencies of eigen-frequency ω on the radial wave number k for fundamental mode $j = 0$ and first four reflective harmonics $j = 1, 2, 3, 4$ are displayed in Fig. 2a-d. Both the pinch-oscillations and the bending modes are unstable ($\text{Im}(\omega) > 0$). The imaginary part of the frequency grows with k and reaches a maximum at some value. The maximum of $\text{Im}(\omega)$ moves to region of shorter wavelengths when the harmonic number j grows, and the value increases with reduction of the characteristic scale of perturbations in z -direction. As indicated by Fig. 2a-d, the growth rate increases with α , while the perturbations phase velocity $\text{Re}(\omega)/k$ does not depend on α . It should be

noted that for very short-wave perturbations ($kh \gtrsim 10$), the damping out of oscillations due to the presence of a perturbed velocity gradient in the viscous stress tensor can be of decisive importance (Khoperskov & Khrapov 1995). This factor can be essential for small-scale waves (as on r - and on z -coordinate) and it can result in complete stabilization.

The vertical structure of the disk in the 3D-model is determined primarily by parameters γ and n . In the case when $n = \gamma$, entropy does not vary along the z -coordinate, i.e. the disk is obviously stable against convective z -motions. When $n > \gamma$ we have $\partial S_0 / \partial z < 0$ and the conditions for convective instability are fulfilled, in the opposite case $n < \gamma$ and $\partial S_0 / \partial z > 0$, so convection can not appear. In Fig. 3a-d, the dispersion curves for various

values of n ($n > \gamma$, $n < \gamma$ and $n = \gamma$) for fundamental modes ($j = 0$) and for the third reflective harmonic ($j = 3$) are shown. The phase velocity of fundamental S- and AS-modes weakly depends on the parameter n (see Fig. 3b,d), as these perturbations are most longwave in z -direction. The adiabatic sound speed in disk C_s grows with n (see Eq. (14)), and since for the high-frequency reflective harmonics $k_z \sim |d\hat{\xi}/\hat{\xi}dz| \neq 0$, the phase velocity of perturbations with $j > 0$ increases also (see Fig. 3b,d).

The dispersion properties of the acoustic perturbations do not depend on the values of parameters l_P and l_ρ in the range $kh > 0.2$. The reason for this is that the radial gradients of equilibrium pressure and density give a small contribution to the equilibrium balance in the case of a thin disk.

5. Discussion and conclusions

Acoustic waves in a differentially rotating gaseous disk can play an important role for understanding the nature of turbulent viscosity in accretion disks. The amplitude of small-scale (in r and in z) waves grows most rapidly. Such instabilities do not destroy the initial flow at a non-linear stage, but can effectively make the disk matter turbulent, and in turn, the arising turbulent viscosity generates unstable sound modes. As a result, a self-consistent regime with turbulent viscosity arises. Beside the dissipational mechanism analysed above, the development of global (covering practically the whole disk radially) resonant Papaloizou-Pringle modes (Papaloizou & Pringle 1987; Savonije & Heemskerk 1990), and also resonant amplification of acoustic oscillations in the regime of double-flow accretion (Mustsevoj & the Khoperskov 1991), and in the case of disk accretion onto a magnetized compact object (Hoperskov et al. 1993) may be important for understanding turbulent viscosity in the disk. It is remarkable that in all cases the characteristic time scale does not exceed the dynamical time scale ($\tau = 1/\text{Im}(\omega) \sim \Omega^{-1}$), and that all values are of the same order.

We have demonstrated the possibility of unstable high-frequency acoustic waves in a differentially rotating gaseous disk. They exist in the system for a limited period, and the presence of a positive growth rate does not mean that the perturbations must reach a non-linear stage. The perturbations leave the disk with a speed $\partial\omega/\partial k \sim C_s kh/\sqrt{1+k^2h^2} < C_s$ and the characteristic life-time in the disk is $\sim r/h \sim 10^2$ disk rotation periods. In view of the derived estimation $\text{Im}(\omega) \sim 0.1\Omega$ for $kh > 1$ we have the essential growth of wave amplitude ($\exp\{0.1r/h\}$). Nonaxisymmetric perturbations may stay in the system for a longer time and reach a non-linear stage.

The following are the main conclusions:

1. The linear wave dynamics of a thin disk model is compared to an exact solution, which takes into account

vertical structure. Our analysis shows that not only the longwave perturbations, but also rather short waves can be considered. A small quantitative difference occurs for a wavelength $\lambda < 2\pi h$, and even in the case of $\lambda > 2\pi h$ the dispersion properties remain qualitatively similar. At $kh \leq 1$ eigenfrequencies differ less than on 5%.

2. In addition to the fundamental dissipational unstable sound mode in MTD, an arbitrary number of high-frequency unstable harmonics in the case of pinch-oscillations is found. These harmonics have a different vertical structure. The waves with small scales along z -coordinate have the maximum growth rate at short wavelengths in the r -direction. The differential rotation and the variable dynamic viscosity cause instability of all oscillation branches, i.e., a dissipational mechanism lead to the perturbation growth.
3. Taking into account the vertical structure of disk we studied new bending modes, which cannot be investigated in the context of MTD. The bending oscillations, as well as the pinch-wave, are unstable. The physical mechanism of instability and the dispersion properties are similar to the case of pinch-oscillations. The significant feature of all considered unstable acoustic modes is the fact that spatial scales of perturbations differ from each other, but characteristic growth rates have the same order of magnitude.

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